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HARMONIC OSCILLATIONS OF AN ELECTROELASTIC SEMI-INFINITE MEDIUM, CAUSED BY A PERIODIC ACTION IN SPACE*

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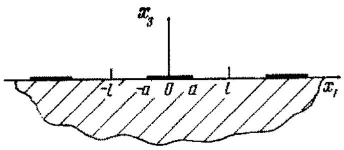
A two-dimensional boundary value problem of the harmonic oscillations of a semi-infinite piezoelectric medium with one flat boundary on which a normal displacement and an electrical field potential are given periodically, is considered. This problem occurs in the design of a number of surface acoustic wave devices /1/. Such devices consist of a piezoelectric crystal of rectangular planform and cross-section on one of whose faces a periodic system of rectangular electrodes is superimposed. The presence of the periodically arranged electrodes on the boundary exerts an influence on the surface acoustic waves by two means: 1) electrical shorting of the surface, and 2) mechanical action on the oscillating medium because of electrode inertia. The contribution of the mechanical action here grows as the operating frequencies of the device increase.

The boundary value problem reduces to a system of periodic convolution equations. The properties of the kernels of the integral equations are established. A theorem is presented that enables one to transfer to the solution of systems of algebraic equations. A solution is constructed for the wave fields at any point of the medium. An example is considered for calculating the wave fields on the boundary of the medium.

1. On the surface of an electroelastic half-space $x_3 \leqslant 0$ adjoining a vacuum, let a system of parallel electrodes, periodic in x_1 with period $2l$ and of width $2a$ be arranged (figure). Within the framework of the electrostatic approximation /2/, the complete system of differential equations has the form /1/



$$c_{ijkl}\frac{\partial^2 u_k}{\partial x_i \partial x_l} + e_{kij}\frac{\partial^2 \varphi}{\partial x_i \partial x_k} = \rho \frac{\partial^2 u_j}{\partial t^2} \tag{1.1}$$
$$e_{ikl}\frac{\partial^2 u_k}{\partial x_i \partial x_l} - \varepsilon_{ik}\frac{\partial^2 \varphi}{\partial x_i \partial x_k} = 0$$

Eqs.(1.1) hold in the domain $x_3 \leqslant 0$. For $x_3 > 0$ the electrical potential is described by the Laplace equation $\Delta\varphi = 0$ /1/.

The boundary conditions are mixed. Conditions on the displacements must be posed on that part of the surface where the electrodes are arranged and the potential must be given, while conditions of no mechanical stresses and continuity of the normal component of the induction vector are given on the free part /3/. Because of the periodicity of the problem it is sufficient to examine one period; the conditions are repeated in the rest. Let the displacements and potential depend on time according to the law $\exp(-i\omega t)$ (we henceforth omit this factor). Assuming the tangential stresses under the electrodes to be small compared with the normal stresses, the following boundary conditions should be satisfied for the two-dimensional problem

$$x_3 = 0,\ x_1 \in [-a,\ a],\ T_{13} = T_{23} = 0,\ u_3 = N_3(x_1),\ \varphi = N_4 \tag{1.2}$$
$$x_3 = 0,\ x_1 \overline{\in} [-a,\ a],\ T_{13} = T_{23} = T_{33} = 0,\ D_3 - D_3{}^b = 0$$

where $D_3{}^b(x_1, x_3)$ is the normal component of the induction in a vacuum.

Let u, v, w denote the displacements u_i $(i = 1, 2, 3)$ in a dimensionless x, y, z coordinate system where $x_1 = lx/\pi,\ x_2 = ly/\pi,\ x_3 = lz/\pi$. To reduce the boundary value problem to a system

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of integral equations, Eqs.(1.1) with the boundary conditions

$$z=0,\ x\in[-\pi,\ \pi],\ T_{13}=T_{23}=0,\ T_{33}=q_1(x),\ D_3-D_3{}^b=q_2(x) \tag{1.3}$$

are considered.

Here $q_1(x)$ and $q_2(x)$ are finite functions equal to zero for $x \overline{\in} [-\alpha,\ \alpha]$, $\alpha = a\pi/l$.

The boundary value problem (1.1) with the boundary conditions (1.3) is reduced to a system of two periodic convolution equations on a segment /4/

$$\int_{-\alpha}^{\alpha} R(x-\xi)\,\mathbf{q}(\xi)\,d\xi = \mathbf{f}(x), \quad x\in[-\alpha,\alpha] \tag{1.4}$$

$$R(t)=\|k_{ij}(t)\|, \quad k_{ij}(t)=\sum_{n=-\infty}^{\infty} K_{ij}(n)\exp(int)$$

$$i,\ j=1,\ 2, \quad \mathbf{q}=(q_1,\ q_2), \quad \mathbf{f}=(N_3,\ N_4)$$

by representing all functions in the form of their Fourier series and later solving the problems that occur for each component.

The functions $K_{ij}(u)$ have the properties:

1) $K_{ii}(u)$ — are even and $K_{12}(u) = K_{21}(u)$ — are odd functions;

2) $K_{ii}(u) = C_{ii}{}^2\,|u|^{-1}\,[1+O(|u|^{-2})]$ $(|u|\to\infty)$, $i=1,\ 2$;

3) $K_{12}(u) = K_{21}(u) = Cu^{-1}\,[1+O(u^{-2})]$ $(u\to\pm\infty)$.

Moreover, the functions $K_{ij}(u)$ $(i,\ j=1,\ 2)$ are fractions with the common even denominator $\Delta(u)$, i.e., $K_{ij}(u) = K_{ij}{}^*(u)/\Delta(u)$. The function $\Delta(u)$ has two real zeros $\pm\zeta$. The following approximations hold ($P_L(u)$ is a polynomial in even powers of L)

$$\Delta(u) = P_L^{\Delta}(u) = (u^2-\zeta^2)\,P_{L-2}^{\Delta}(u)$$

$$K_{ii}{}^*(u) = P_L{}^{ii}(u),\ i=1,\ 2;\ K_{12}{}^*(u) = K_{21}{}^*(u) = P_{L-1}^{12}(u)$$

A basis for the introduction of such an approximation is to take account of a finite number of wave modes, starting with the lowest, that occur in an electroacoustic medium as the mechanical and electrical effects vary harmonically.

The functions $K_{ij}(u)$ can be represented in the form

$$K_{ii}(u)=\frac{A^{ii}}{A^{\Delta}}\prod_{j=1}^{L}(u-z_j{}^{ii})\Big[\prod_{j=1}^{L}(u-\zeta_j)\Big]^{-1}=\frac{A^{ii}}{A^{\Delta}}+ \tag{1.5}$$

$$\frac{P_{L-1}^{ii}(u)}{P_L{}^{\Delta}(u)}, \quad K_{12}(u)=K_{21}(u)=\frac{P_{L-1}^{12}(u)}{P_L{}^{\Delta}(u)}; \quad \zeta_{L-1}=\zeta, \quad \zeta_L=-\zeta$$

Here ζ_j $(j=1,\ldots,L-2)$ are zeros of the polynomial $P_{L-2}^{\Delta}(u)$; $z_j{}^{ii}$ are the zeros of the polynomials $P_L{}^{ii}(u)$, and A^{ii}, A^{Δ} are coefficients of the highest powers of the corresponding polynomials, $i=1,\ 2$.

2. Let us consider the system of integral Eqs.(1.4) in which $K_{ij}(n)$ are written in the form (1.5).

Theorem. For a function of the form $\Phi(z) = K(z)\exp(izt)$ for which $|t| < 2\pi$, $|K(z)|\to C\,|z|^{-1}$ as $|z|\to\infty$; $K(z)$ has the poles ζ_j $(j=1,\ldots,L)$ the representation

$$\sum_{n=-\infty}^{\infty}\Phi(n) = -\sum_{j=1}^{L}\operatorname{Res}(\Phi(z)\operatorname{ctg}\pi z)|_{z=\zeta_j} + \frac{t}{|t|}\pi i\sum_{j=1}^{L}\operatorname{Res}\Phi(z)|_{z=\zeta_j}$$

holds.

The theorem is proved by the same scheme as Theorem 1.4.1 in /5/ except the presence of the poles of the function $K(z)$ must be taken into account by applying theorems on residues.

System (1.4) with kernels in which the functions $K_{ij}(u)$ have the form (1.5) is reduced to the system

$$2\pi\,(A^{ll}/A^{\Delta})\,q_l(x) + I_{1l}(x) + I_{2l}(x) = f_l(x),\ x\in[-\alpha,\ \alpha] \tag{2.1}$$

$$I_{kl}(x) = \int_{-\alpha}^{x} q_k(\xi)\sum_{j=1}^{L} R_j^{lk}\theta_j(x-\xi)\,d\xi + \int_{x}^{\alpha} q_k(\xi)\sum_{j=1}^{L} H_j^{lk}\theta_j(x-\xi)\,d\xi$$

$$R_j^{kl} = D_j^{kl} + S_j^{kl}, \quad H_j^{kl} = D_j^{kl} - S_j^{kl}, \quad \theta_j(x-\xi) = \exp[i\zeta_j(x-\xi)]$$

by using the theorem, where D_j^{kl} are residues of the function $\operatorname{ctg} \pi u P_{L-1}^{kl}(u)/P_L^{\Delta}(u)$ taken at the points $u = \zeta_j$, and S_j^{kl} are residues of the function $i\pi P_{L-1}^{kl}(u)/P_L^{\Delta}(u)$, $k, l = 1, 2$.

The function $f_1(x)$ in the system (2.1) is the normal electrode displacement. Let it be given by a Fourier series; we solve system (2.1) for the arbitrary component $F_1 \exp(i\eta x)$ of this series. The function $f_2(x)$ is the magnitude of the potential on the electrode, consequently, we set it equal to the constant F_2. We solve system (2.1) with the right side

$$f_1(x) = F_1 \exp(i\eta x), \quad f_2(x) = F_2 \tag{2.2}$$

The solution of system (2.1) with the right side (2.2) is sought in the form

$$q_i(\xi) = A_i + B_i \exp(i\eta\xi) + \sum_{k=1}^{N} X_k^{(i)} \exp(ip_k\xi) \tag{2.3}$$

where $A_i, B_i, X_k^{(i)}$ $(i = 1, 2;\ k = 1, \ldots, N)$ are unknown coefficients, p_k $(k = 1, \ldots, N)$ are also unknown quantities the number and values of which are determined during the solution. After substituting (2.3) into (2.1) with the right side (2.2) and evaluating the integrals in each equation of (2.1), components appear that are independent of x, components with the factors $\exp(i\eta x)$, with the factors $\exp(ip_k x)$, and also with the factors $\exp(i\zeta_j x)$ $(j = 1, \ldots, L)$. Because of the linear independence of the components with the mentioned factors, the system of integral Eqs. (2.1) uncouples into a number of linear algebraic systems

$$M(0)\cdot\begin{Vmatrix} A_1 \\ A_2 \end{Vmatrix} = \begin{Vmatrix} 0 \\ F_2 \end{Vmatrix}, \quad M(\eta)\cdot\begin{Vmatrix} B_1 \\ B_2 \end{Vmatrix} = \begin{Vmatrix} F_1 \\ 0 \end{Vmatrix} \tag{2.4}$$

$$M(p_k)\cdot\begin{Vmatrix} X_k^{(1)} \\ X_k^{(2)} \end{Vmatrix} = \begin{Vmatrix} 0 \\ 0 \end{Vmatrix} \tag{2.5}$$

$$M^{(1)}\cdot\mathbf{X}^{(1)} + M^{(2)}\cdot\mathbf{X}^{(2)} = M^P\cdot(A_1, A_2, B_1, B_2)^T \tag{2.6}$$

$$\mathbf{X}^{(i)} = (X_1^{(i)}, \ldots, X_N^{(i)}), \quad i = 1, 2$$

Here $M(\eta)$ is a (2×2) matrix whose elements depend on the parameter η, $M^{(1)}$, $M^{(2)}$ are $(N \times N)$ matrices, and M^P is a $(2L \times 4)$ matrix.

Here A_1, A_2 and B_1, B_2 are determined from (2.4). For a non-zero solution of system (2.5) it is necessary to satisfy the condition

$$|M(p_k)| = 0 \tag{2.7}$$

Eq. (2.7) is a polynomial of degree $2L$, consequently, there are $2L$ roots p_k for which the non-zero pairs $X_k^{(1)}$, $X_k^{(2)}$ exist that satisfy system (2.5). Therefore, $N = 2L$ in (2.3). It is possible to express $X_k^{(1)} = \Phi_k X_k^{(2)}$ from (2.5). Then the unknowns $X_k^{(2)}$ are determined from the inhomogeneous system (2.6) of N linear algebraic equations in the N unknowns $X_k^{(2)}$ $(k = 1, \ldots, 2L = N)$, and functions of the mechanical stress $q_1(x)$ and the charge density $q_2(x)$ are constructed by means of (2.3). Formulas (2.1)-(2.4) in /4/ enable the displacements $u(x, z)$, $v(x, z)$, $w(x, z)$ and the potential $\varphi(x, z)$ to be determined for known $q_1(x)$ and $q_2(x)$ in the domain

$$-\infty \leqslant x \leqslant \infty, \ z \leqslant 0.$$

3. As an illustration, calculations were made for the displacements and the potential on the boundary of an electroelastic quartz crystal half-plane with a periodic system of electrodes. The half-plane boundary belongs to the *ST*-cut quartz surface. The computations were performed for the simplest approximation $L = 2$ for the following parameters $\omega = 20\pi$ MHz $l = 0{,}1579 \times 10^{-3}$ m, $\eta = 1$, $F_1 = 0{,}194 \times 10^{-11}$ m, $F_2 = 0$. i.e., it is assumed that the lattice equals the Rayleigh wavelength on the free *ST*-cut quartz surface and all the electrodes are grounded. The dimensionless coordinate α of the electrode edge was varied in steps of 0,1 between 0,1 and 3,1 and the functions $w(x, 0)$ and $\varphi(x, 0)$ were evaluated for each fixed α. It was established that the amplitudes of the displacement $w(x, 0)$ and the potential $\varphi(x, 0)$ reach the maximum values for $\alpha = 0{,}6$. One should start from this quantity in designing such devices as resonators where it is necessary to reach the maximum amplitudes at a given frequency.

4. The method elucidated for solving a periodic problem for an electroelastic half-plane also permits the solution of a more general problem without the assumption that the shear stress under the electrodes is small compared with the normal stresses. The two-dimensional boundary value problem is here reduced to a system of three integral equations of the form (1.4), system (2.1) is correspondingly more complicated, and the number and dimensionality of the algebraic systems (2.4)-(2.6) is also increased.

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APPLICATION OF STRUCTURAL REPRESENTATIONS TO THE SOLUTION OF BOUNDARY-VALUE PROBLEMS OF IDEAL PLASTICITY*

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The method of successive approximations is proposed for solving plane problems of the theory of ideal plasticity, based on structural representations. By using this method the state of stress is determined in an infinite plane with a circular hole for an arbitrary change in the forces applied at infinity. It is shown that for a certain constraint on the asymmetry of the loads, the solution of the problem considered is independent of the loading trajectory. When the mentioned constraint is violated, the plasticity domain will be different depending on the history of the load change.

1. The stresses caused by inelastic strain $\Gamma_{ij}\ (i, j = x, y, z)$ can be represented in the plane case /1/ as stresses due to wedgelike dislocations (WD), distributed (inserted) over the plasticity domain (PD) with density $p(x, y)$ and over its boundary L with density $p_L(l)$. The magnitudes of the densities under plane strain have the form

$$p(x, y) = \frac{\partial^2\Gamma_x}{\partial y^2} + \frac{\partial^2\Gamma_y}{\partial x^2} - 2\frac{\partial^2\Gamma_{xy}}{\partial x\partial y} + \nu\Delta\Gamma_z \tag{1.1}$$
$$p_L(l) = \left(\frac{\partial\Gamma_{xy}}{\partial y} - \frac{\partial\Gamma_y}{\partial x}\right)\cos(nx) + \left(\frac{\partial\Gamma_{xy}}{\partial x} - \frac{\partial\Gamma_x}{\partial y}\right)\cos(ny) - \nu\frac{\partial\Gamma_z}{\partial n}$$

($\Gamma_i = \Gamma_{ii}$, ν is Poisson's ratio, and n is the external normal to the boundary of the plasticity domain).

In the plane state of stress the values of the densities are obtained from the expression presented if the last terms dependent on the strain Γ_z are discarded.

On the other hand, if the state of stress of a body is known, it completely determines those structural imperfections that were formed therein at the time under consideration. The density of these imperfections (i.e., the WD) in the PD and on its boundary is determined by the expressions /1/

$$p(x, y) = -\frac{1+\varkappa}{8G}\Delta(\sigma_x + \sigma_y) \tag{1.2}$$

$$p_L = -\frac{1+\varkappa}{8G}\left[\frac{\partial(\sigma_x + \sigma_y)}{\partial n}\right]_L \tag{1.3}$$

The square brackets here denote discontinuities of the quantities therein on the inelastic strain domain boundary; it is calculated for the passage from points within the domain

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